Toric Varieties for Skeptics
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Cottbus, September 2025

Exercises & Examples 1

Recall the definition of a *smooth polytope*. Draw a smooth polygon P.

Exercise 1.1

Write down numbers next to the lattice points of P which define a $\mathbb{Q}$-valued point p of X_P. Express your point in the (affine) coordinates corresponding to different vertices of P. Can you find a general formula for this coordinate change?

Find a vertex where p does not belong to the corresponding affine chart. If you cannot find such a vertex, pick a new point $q \in X_P$ so that you can.

Repeat with a different polygon.

Exercise 1.2

Specify a torus element $t \in T \circlearrowright X_P$, and determine $t.p \in X_P$.

For a face F of P pick two points p and q from the corresponding T-orbit. Then find an element $t \in T$ so that $t.p = q$. Repeat for faces of different dimensions.

For an edge F of P choose $p \in T$ and q in the T-orbit corresponding to F. Can you find a sequence $(t_k)_{k\geq 0}$ in T so that $\lim_{k\to\infty} t_k.p = q$?

Exercise 1.3

Choose a lattice basis for M and describe the torus action in coordinates at different vertices.

Exercise 1.4

We have two different descriptions of X_P: One as a space glued from affine spaces, one for each vertex, and the other description as a subset of projective space.

What could go wrong if we drop the smoothness requirement? In other words, what property is needed to ensure that the two spaces agree?

Convince yourself that it never goes wrong for polygons, but it does go wrong for

$$P = \operatorname{conv} \begin{bmatrix} 0&1&0&0&1&0&1&1 \\ 0&0&1&0&0&1&1&1 \\ 0&0&0&1&1&1&8&9 \end{bmatrix} .$$

Exercise 1.5

Prove that you criterion from the previous exercise is indeed a characterization.

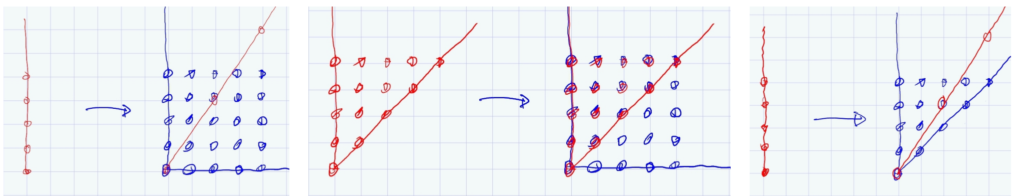
Exercise 1.6

Consider two lattices $N \subset V$, $N' \subset V'$, and two pointed rational cones $\sigma \subset V$, $\sigma' \subset V'$ with polar duals $\sigma^\vee \subseteq V^*$, $\sigma'^\vee \subseteq V'^*$, respectively. Suppose $\Phi\colon N \to N'$ is linear so that $\Phi(\sigma) \subseteq \sigma'$. Use the transpose $\Phi^t\colon M' \to M$ to define induced maps

$$\begin{array}{ccc} X_{\sigma^\vee} & \xrightarrow{\Phi^t} & X_{\sigma'^\vee} \\ \circlearrowright & & \circlearrowright \\ T & \xrightarrow{\Phi^t} & T' \end{array}$$

Exercise 1.7

Find the maps induced by the following Φ's.



Over the field $\mathbb{R}$, you should get maps $\mathbb{R} \to \mathbb{R}^2$, $\mathbb{R}^2 \to \mathbb{R}^2$, and $\mathbb{R} \to \mathbb{R}^2$. Draw a picture of the two parameterized curves $\mathbb{R} \to \mathbb{R}^2$.

Exercise 1.8

Interpret an integral linear map $P \twoheadrightarrow [0, \ell]$ as a family of rational curves of degree ℓ

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & X_P \\ & & \downarrow \\ & & B \end{array}$$

in X_P over a base B of dimension $\dim X_P - 1$. Interpret the lattice width of P in this language.

Find a P and a rational curve $\mathbb{P}^1 \hookrightarrow X_P$ of degree strictly smaller that P's lattice width. Can your curve move in a family?

Exercise 1.9

Let $f \in \mathbb{k}[x, y, z]$ be an irreducible homogeneous polynomial of degree D, and consider its vanishing locus $X \subset \mathbb{P}^2$. Then the homogeneous coordinate ring $R = \mathbb{k}[x, y, z]/(f)$ of X is graded: $R = \bigoplus_{d \geq 0} R_d$. Show that $\dim_{\mathbb{k}} R_d \sim D \cdot d$ for $d \to \infty$, and convince yourself that $D = \#(X \cap H)$ where $H \subset \mathbb{P}^2$ is a general hyperplane (aka line).

Exercise 1.10

Suppose $X \subset \mathbb{P}^{N-1}$ is an irreducible subvariety of dimension n with homogeneous coordinate ring $R = \bigoplus_{d \geq 0} R_d$. Show that there is an integer D so that $n! \cdot \dim_{\mathbb{k}} R_d \sim D \cdot d^n$ for $d \to \infty$.

Convince yourself that $D = \#(X \cap U)$ where $U \subset \mathbb{P}^{N-1}$ is a general subspace of codimension n.

Exercise 1.11

Given lattice polytopes $P_1, \dots, P_d \subset \mathbb{R}^d$, consider general (Laurent-)polynomials $f_1, \dots, f_d \in \mathbb{k}[x_1^{\pm 1}, \dots x_d^{\pm 1}]$. These will have finitely common zeros in $(\mathbb{k}^\times)^d$. Call the number of these zeros $M(P_1, \dots, P_d)$. (You have to believe that the number does not depend on the f's as long as they are generic enough and $\mathbb{k} = \bar{\mathbb{k}}$.)

Show that $M(P_1 + P_1', P_2, \dots, P_d) = M(P_1, \dots, P_d) + M(P_1', \dots, P_d)$.

Exercises & Examples 2

Exercise 2.1

Let $I \subset \mathbb{k}[x_1, \dots, x_N]$ be an ideal, $J = \mathrm{in}_\omega(I)$ a monomial initial ideal with Gröbner basis $\mathcal{G}$ and standard monomials S (monomials not in J). Then polynomial division writes every $f \in \mathbb{k}[x_1, \dots, x_N]$ modulo $\mathcal{G}$ as a $\mathbb{k}$-linear combination of monomials in S.

Show that this induces an isomorphism $\mathbb{k}[x_1, \dots, x_N]/I \to \mathbb{k}^S$ of $\mathbb{k}$-vector spaces.

Exercise 2.2

Show $x^3 - y^2 \in (x - y, x^2 - y)$.

Exercise 2.3

Find as many equations as possible satisfied by $X_P \subset \mathbb{P}^5$ for the rectangle $P = \operatorname{conv} \left[\begin{smallmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right]$. Did you find enough to generate the ideal of all such polynomials?

Write down a $t \in T$ as in Exercise 1.2 and convince yourself that $t.x$ satisfies your equations whenever x does.

Exercise 2.4

Find as many equations as possible satisfied by $X_P \subset \mathbb{P}^3$ for the segment $P = \operatorname{conv}[0, 3]$, so $\mathcal{A} = \left[\begin{smallmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right]$. Find two general weights $\omega, \omega' \in \mathbb{R}^{\mathcal{A}}$ which induce the same triangulation $\mathrm{sd}_\omega(\mathcal{A}) = \mathrm{sd}_{\omega'}(\mathcal{A})$ of P, but yield different initial ideals $\mathrm{in}_\omega(I_{\mathcal{A}}) \neq \mathrm{in}_{\omega'}(I_{\mathcal{A}})$.

We can restrict to $\omega_0 = \omega_3 = 0$. Then, draw a picture of the regions of the ω_1-ω_2-plane where $\mathrm{sd}_\omega(\mathcal{A})$ is constant. Do the same for the regions where

Exercise 2.5

Revisit Exercise 2.1 in the examples of Exercise 2.4.

Exercise 2.6

For $P = \operatorname{conv} \left[\begin{smallmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{smallmatrix}\right]$ show that the corresponding toric ideal $I_{\mathcal{A}}$ is not generated in degree 2.

Exercise 2.7

Let P be a lattice polygon with at least 4 boundary lattice points. Show that $I_{\mathcal{A}}$ is quadratically generated. Show that it even has a quadratic Gröbner basis.

Exercise 2.8

Given natural numbers m, n, let $\mathcal{A} \in \mathbb{R}^{mn \times m+n}$ be the matrix which sends a matrix $A \in \mathbb{R}^{mn}$ to its row and column sums in $\mathbb{R}^{m+n}$.

Show that the toric ideal $I_{\mathcal{A}}$ is generated by moves of the form

$$\begin{bmatrix} & \vdots & & \vdots & \\ \cdots & 1 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & -1 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \end{bmatrix}$$

Exercise 2.9

Let $\mathcal{A}$ be the indicator vectors of the bases of a matroid M. We call M quadratic if $I_{\mathcal{A}}$ is quadratically generated. Reformulate being quadratic as an exchange axiom in combinatorial matroid language.

Can you find a matroid which is not quadratic?